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ANALYTICALLY COMPUTED RATES OF SEEPAGE FLOW INTO DRAINS AND CAVITIES

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SUMMARY

The known formulae of Freeze and Cherry, Polubarinova–Kochina, Vedernikov for flow rate during 2-D seepage into horizontal drains and axisymmetric flow into cavities are examined and generalized. The case of an empty drain under ponded soil surface is studied and existence of drain depth providing minimal seepage rate is presented. The depth is found exhibiting maximal difference in rate between a filled and an empty drain. 3-D flow to an empty semi-spherical cavity on an impervious bottom is analysed and the difference in rate as compared with a completely filled cavity is established. Rate values for slot drains in a two-layer aquifer are 'inverted' using the Schulgasser theorem from the Polubarinova–Kochina expressions for corresponding flow rates under a dam. Flow to a point sink modelling a semi-circular drain in a layered aquifer is treated by the Fourier transform method. For unsaturated flow the catchment area of a single drain is established in terms of the quasi-linear model assuming the isobaric boundary condition along the drain contour. Optimal shape design problems for irrigation cavities are addressed in the class of arbitrary contours with seepage rate as a criterion and cavity cross-sectional area as an isoperimetric restriction.

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Key words: seepage; flow rate; drain; tunnel; optimization

1. INTRODUCTION

Drains, openings, tunnels or natural cavities are either designed for purposive seepage or are influenced by undesirable subsurface flows in their vicinity. In any case, accurate prediction of total volume of seeping water and hydraulic gradients is important. Standard software may fail to reproduce these characteristics in a 'fine scale' and analytic expressions can be useful.

With the advent of FDM–FEM packages analytic solutions for seepage flow problems became a supplementary tool in engineering practice. However, new environment like *Mathematica* allows one to reconsider applicability of 'old-fashioned' analytic technique which restores from seemingly ponderous forms into standard built-in computer operations. In this paper, we study analytically seepage into horizontal drains and cavities. Our goal is both to fill in the lacunae in the classical books[1–5] – and to derive some new solutions. We focus our interest on one of the most important characteristics, total rate of water seeping into a drain, even though other

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distributed (flow nets, specific discharge, moisture, etc.) or integral (erosion safety factors) characteristics can be analysed in a similar way. For some specific flow patterns we answer the following questions: What is the difference in seepage rate into an empty and water filled drain? Is there an optimal tunnel depth providing minimal rate for ponded conditions? How strong is the influence of soil heterogeneity (layering) on the value of rate? What is the influence of cavity shape on the rate and is there an optimal form providing minimal rate under imposed isoperimetric restrictions? What is the difference in rate for saturated and unsaturated flows? How do rates differ for 2-D and 3-D patterns? How accurate a real drain or opening can be modelled by a 2-D or 3-D sinks?

Review is out of the scope and limited space of this paper and we reference only few related works. We assume steady Darcian flows of incompressible one-phase fluids in rigid porous media.

2. PONDED SEEPAGE INTO AN EMPTY DRAIN

Consider an empty drain $ABCA$ of radius r located in a homogeneous, isotropic half-plane at the depth c under the soil surface a–a ponded by water with depth H (Figure 1(a) shows a cross-section). In the flow domain the hydraulic head $h(x, y)$ satisfies the Laplace equation. We are

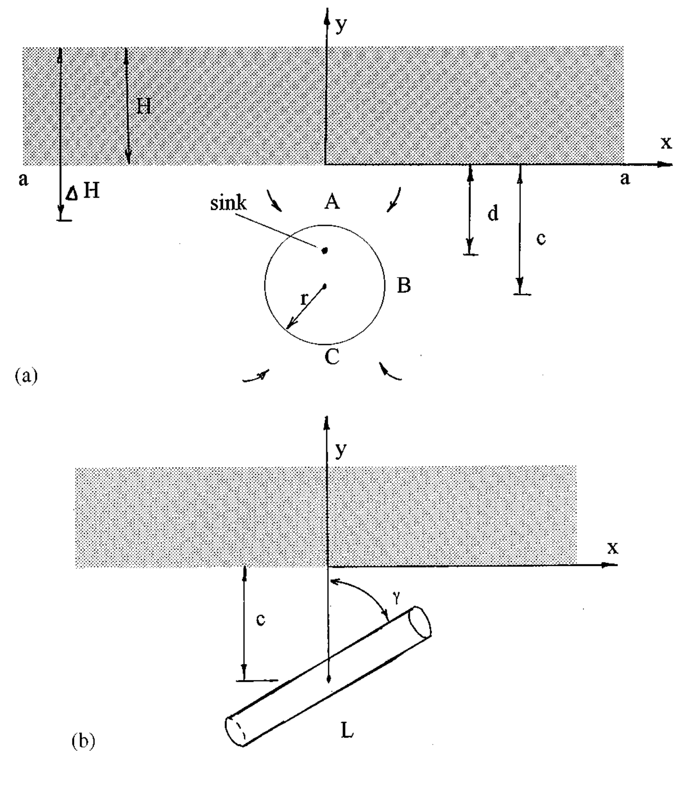


Figure 1

interested in the value of total seepage rate q_e into the drain and its difference from the rate q_f of a drain filled with water.

The case of a filled drain is described everywhere and the formula obtained by Forchheimer[2] in 1889 states

$$q_f = \frac{2\pi\Delta H}{\operatorname{arc\,cosh}(c/r)} \tag{1}$$

where ΔH is the head difference between a–a and the drain contour. Polubarinova-Kochina[4] (p. 354) gives an approximate formula $q_f = 2\pi\Delta H/\ln(2c/r)$. Here and below for saturated homogeneous matrixes we normalize ϕ and q to hydraulic conductivity k. Note, that Polubarinova-Kochina reproduced only an approximation of the exact formula of Vedernikov[5], who repeated the formula of Forchheimer.

For the sake of further comparison with empty drains restrict ourselves (without any loss of generality) with the limiting case when the pressure p (divided by the specific weight of water) at the point A (drain top) is atmospheric (zero), i.e. $\Delta H = H - y_A$. Thorough explanation of boundary conditions along a drain contour is given by Khan and Rushton.[6]

Forchheimer modelled the drain (well) by a point sink placed at the depth $d = c\tanh(2\pi\Delta H/q_f)$. Note that the Forchheimer drain contour (an equipotential of the corresponding sink) is exactly a circle while for all other flow patterns[1,5] equipotentials deviate from circles.

For the case of empty drain (tunnel) Freeze and Cherry[3] say (p. 490, equation (10.17) that the only theoretical formula of Goodman *et al*[7]. for the rate is $q_e = 2\pi\Delta H/[2{\cdot}3\ln(2c/r)]$ (this statement was reproduced in the karst literature[8]). Lei[9] showed that this formula is wrong and emphasized the priority of Polubarinova-Kochina in studies of filled drains.

To derive the correct formula for an empty drain we utilize the 'sink-solution' (Polubarinova-Kochina,[4]) p. 353):

$$\phi(x, y) = -h = \frac{q}{4\pi}\ln\frac{x^2 + (y-d)^2}{x^2 + (y+d)^2} \tag{2}$$

where ϕ is the velocity potential, q and d are the sink strength and depth, respectively. Set $\phi = 0$ along $a - a$. Then along the contour of an empty drain $p = \phi + y - H$. In contrast with the case of a filled drain we search not for an equipotential, but for an isobar $p = 0$ as the drain contour. According to (2) we find the co-ordinates y_A and y_C of points A and C, i.e. the roots of the equations

$$\frac{y-d}{y+d} = \pm E, \qquad E = e^{[2\pi(H-y)]/q} \tag{3}$$

The centre of the drain is located at the point $c = -(y_A + y_C)/2$. Obviously, the isobar $p = 0$ with contour equation

$$x(y) = \pm\sqrt{\frac{E^2(y+d)^2 - (y-d)^2}{1-E}} \tag{4}$$

is not exactly a circle. However, for wide range of the problem parameters (q, d, H) deviation of (4) from a circle is small. Hence, according to Polubarinova-Kochina[4] we derive the radius of an

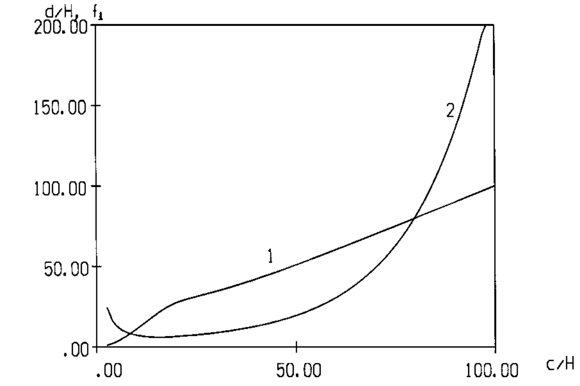


Figure 2

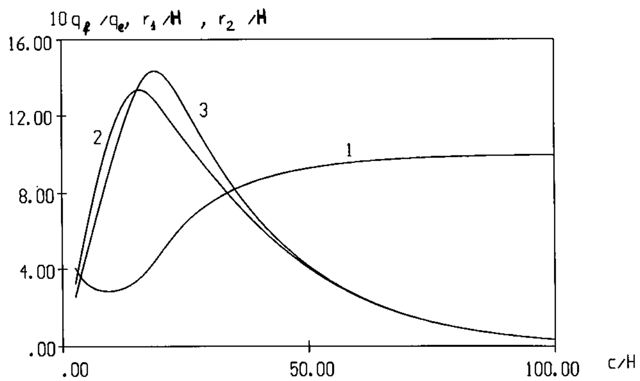


Figure 3

equivalent circular drain as $r_1 = \sqrt{S/\pi}$ where S is the cross-sectional area confined by the curve $ABCA$

$$S = 2\int_{y_C}^{y_B} x(y)\,\mathrm{d}y \tag{5}$$

Another definition of the drain radius is $r_2 = (y_A - y_C)/2$. After expression of c and r in terms of q, d, H we can regard q as q_e.

Figure 2 illustrates the dependence of d/H (curve 1) and $f_1 = 2q_e/\sqrt{S}$ (curve 2) on c/H plotted for $q/H = 100$. The second curve exhibits a minimum $f_1 = 5{\cdot}97$ at $c/H = 15{\cdot}75$. It is clear because q_e tends to infinity at $\Delta H \to 0,\ \infty$ (and $H \neq 0$). Hence, the depth providing this minimum has to exist. Figure 3 presents the graphs of $10q_f/q_e$ (curve 1), r_1/H (curve 2) and r_2/H (curve 3) as functions of c/H. The first curve shows that maximal influence of water level in the drain (minimal value of q_f/q_e is 0·29) occurs at $c/H = 9{\cdot}29$. In other words, at this depth influence of the boundary condition along the drain contour on the rate is most pronounced. Obviously, if the drain is

partially filled its rate q_p can be estimated[10] as $q_f < q_p < q_e$. The last two curves in Figure 3 show that both definitions of the radius (r_1 and r_2) of our drain provide close results, especially at sufficiently high c/H.

Note, that Lei[9] treated analytically the case of a circle even though his regime differs from ours. Namely, for an empty cavity he considered a descendant seepage which is partially catched by a cavity and partially flows down such that flow velocity is constant at infinity. For our case velocity decreases to zero sufficiently far from the cavity. Emphasize, that deviation of (4) from a circle should be checked for any specific set of parameters.

3. 3-D SEEPAGE TO DRAINS AND CAVITIES

3-D fluid flows were widely studied in petroleum engineering[11] and subsurface hydrology.[4] In contrast with broad variety of 2-D schemes treated analytically by the method of conformal mappings, for 3-D flows explicit solutions were derived only for simplest geometries (allowing for separation of variables in the Laplace equation). In what follows, we derive one of these solutions for flow into an empty hemispherical cavity. But, first we list the known formulae for 3-D seepage rate q of filled cavities dividing the rate by the surface area S of a cavity as $\mu = q/S$.

Flow into a spherical cavity-equipotential of arbitrary radius r whose centre is placed under a ponded surface a–a at the depth c is explicitly derived in bi-polar coordinates and the corresponding solution can be found in many textbooks on special functions and electrostatics. Normalized total rate is

$$\mu_{sp} = \frac{0{\cdot}5\Delta H}{r}\left[1 + \frac{e(4 - e^2)}{8 - 4e - 2e^2}\right] \tag{6}$$

where $e = r/c$. For a cavity of small radius ($r \ll c$) the draining sphere can be modelled by a 3-D point sink (Polubarinova-Kochina,[4] p. 366) placed at $y = -c$. It can be shown that after some algebra

$$\mu_{si} = \frac{\Delta H}{r}(1 + e/2) \tag{7}$$

where effective radius r is derived similarly to the case of a 2-D sink.

Figure 4 shows the graphs of μ_{sp} (curve 1) and μ_{si} (curve 2) as functions of $c/\Delta H$ at $r/\Delta H = 1$ calculated according to (6) and (7). These curves shows that sink approximation is perfect for large $c/\Delta H$.

For a cylinder of radius r and length L placed under ponded surface at the depth c and the angle γ between the cylinder axis and the vertical coordinate axis y (Figure 1(b)) the rate is derived approximately by Polubarinova-Kochina[4] (pp. 376–380). She employed the method of distributed sinks and obtained the formula:

$$q_c = \frac{2\pi\Delta HL}{E}, \qquad E = \ln\frac{\mathrm{L}}{\mathrm{r}} - 0{\cdot}5\ln\frac{\sqrt{e^2 + 16 + 8e\cos\gamma} + 4\cos\gamma + e}{\sqrt{e^2 + 16 + 8e\cos\gamma} + 4\cos\gamma - e} \tag{8}$$

where $e = L/c$. For $\gamma = 0$, $\pi/2$ this formula is reduced to the known solution for a vertical and horizontal well under ponded surface (see equivalent formulations for antennae[12]). These limiting formulae are valid for $r \ll L$[12]. Kutateladse[13] imposes additional restrictions on radius, screen length and depth of the cylinder: $c \geqslant 2{\cdot}5r$ (a horizontal well) and $2c - L \geqslant 5$ (a vertical well).

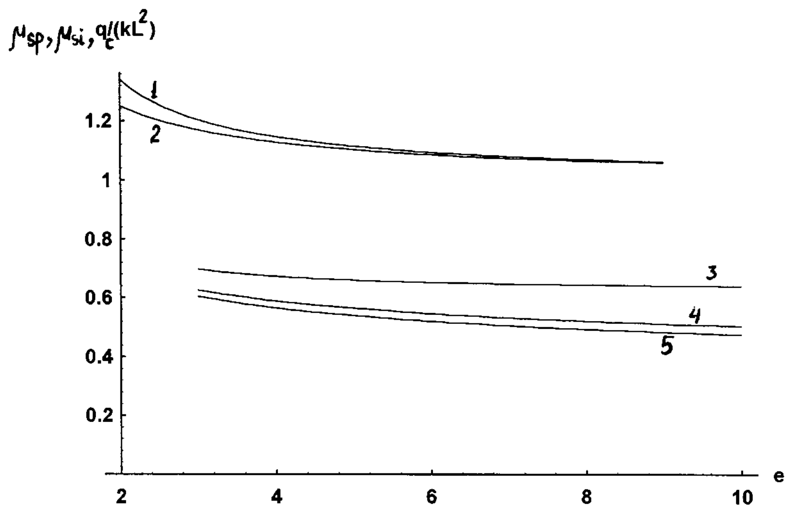


Figure 4

Curves 3–5 corresponding to $\gamma = 0, \pi/3, \pi/2$ in Figure 5 show the rate q_c/L^2 as a function of e calculated according to (8) for $\Delta H/L = 0{\cdot}3$ and $r/c = 0{\cdot}1$. From these graphs we can asses how the flow rate to a cylinder varies with its rotation about the centre.

Let us consider now a hemispherical cavity ABC of radius r placed on an impervious bottom and draining an upper semi-space. Figure 5 shows an axial section, AN and CN are no-flow lines. Assume that at infinity $\phi = -H$.

For a filled cavity (ABC being an equipotential line $\phi = 0$) the normalized rate, $v = q/\sqrt{S}$, is $v_s = \sqrt{2\pi}H$. It is noteworthy, that for the Weber disk (Crank,[14] pp. 42–43) of the same radius (i.e. a circular hole in the bed of an aquifer) $v_w = 4/\sqrt{\pi}H$.

At the points A, C of an empty cavity, set potential $\phi = 0$. Pressure is atmospheric along ABC and, hence, here we have the standard seepage face condition $\phi = -y$. Obviously, for an empty cavity we have to assume $r \ll H$, i.e. the head in the aquifer sufficiently far from the cavity remains undisturbed by seepage to this opening. Introduce the function $\phi^* = \phi + H$ which satisfies the Laplace equation in the flow domain, vanishes at infinity and along the cavity surface is $\phi^* = H - y$ (we drop further the stars). It is well-known that in the this case a harmonic function can be represented as a series of the Legendre polynomials of first kind:

$$\phi(\rho, \theta) = \sum_{k=0}^{\infty} b_k \rho^{-k-1} P_k(\cos\theta) \tag{9}$$

where $r \leqslant \rho < \infty$, $0 \leqslant \theta < \pi/2$ are spherical coordinates and b_k are coefficients to be found. Note, that generally (9) should involve another series of the Legendre polynomials of second kind, but for our specific problem it is easy to show that corresponding coefficients are zero. Recall the analogy with 2-D solutions involving seepage face conditions and Chebyshev polynomial expansions.[15]

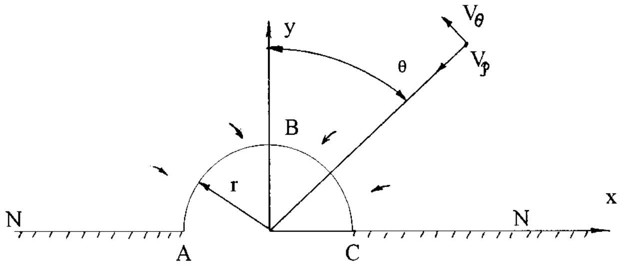


Figure 5

The radial and tangential components of specific discharge (normalized to conductivity) are

$$v_\rho = \frac{\partial\phi}{\partial\rho} = -\sum_0^\infty b_k(k+1)\rho^{-k-2}P_k(\cos\theta),$$

$$v_\theta = \frac{\partial\phi}{\rho\partial\theta} = -\sum_0^\infty b_k k\rho^{-k-2}\frac{P_{k-1}(\cos\theta) - \cos\theta\, \mathrm{P}_k(\cos\theta)}{\sin\theta} \tag{10}$$

At $\theta = \pi/2$ the tangential velocity is zero and, hence, from the second expression in equation (10) $b_{2k-1} = 0$. Even coefficients are defined from the boundary condition along the seepage face. Namely, along ABC

$$\sum_0^\infty b_{2k}r^{-2k-1}P_{2k}(\tau) = H - r\tau \tag{11}$$

where $\tau = \cos\theta$, $0 < \tau < 1$. We have to expand the right-hand side in (11) in a series of even Legendre polynomials. Multiply both sides of (11) successively by $P_{2m}(\tau)$, $m = 0, 1, 2, \ldots$ and integrate from 0 to 1. Then we come to the system

$$b_{2k}r^{-2k-1}\int_0^1 P_{2k}(\tau)P_{2m}(\tau)\,\mathrm{d}\tau = H\int_0^1 P_{2m}(\tau)\,\mathrm{d}\tau - r\int_0^1 P_{2m}(\tau)\tau\,\mathrm{d}\tau \tag{12}$$

By virtue of orthogonality, the first integral in (12) equals zero at all $k \neq m$ (Reference 16, formula 1, Section 2.17.14) and at $k = m$,

$$\int_0^1 P_{2k}^2(\tau)\,\mathrm{d}\tau = \frac{1}{4k+1}$$

The second integral in (12) is zero at $m = 1, 2, \ldots$ and equals 1 at $m = 0$ (Reference 16, 1983, formula 2, Section 2.17.1). The third integral (Reference 16. 1983, formula 1, Section 2.17.1) is

$$J_{2k} = \int_0^1 P_{2k}(\tau)\tau\,\mathrm{d}\tau = \frac{(-1)^k(-1/2)_k}{2(1)_{k+1}} = \frac{(-1)^{k+1}(2k-3)!!}{2^{k+1}(k+1)!}$$

where $(\ )_n$ designates the Pochhammer symbol (expression through factorials is given only for illustration because the Pochhammer symbol as well as the orthogonal functions are built-in

procedures in *Mathematica*). Now, we substitute the integrals calculated into equation (12) that yields

$$b_0 = rH - r^2/2, \qquad b_{2k} = (4k+1)J_{2k}r^{2k+2}, \qquad k = 1, 2, \ldots \tag{13}$$

Substitute the coefficients found into the first equation (9). Then at $\rho = r$ we derive the distribution of normal velocity along ABC which is used in estimations of slope stability during seepage. Figure 6 shows $v_\rho(\theta)$ computed at $r/H = 0{\cdot}1$. We retained 300 terms in the series and used the asymptotic formula for the Legendre functions at $k > 7$: $P_k = \sqrt{2/(k\pi \sin\theta)}\,\sin[(k+1/2)\theta + \pi/4]$. These asymptotics are valid for $\theta_0 \leqslant \theta$, i.e. not too close to the point B. Comparisons with the rigorous Legendre presentations showed good agreement between them and the asymptotic expression.

Obviously, the seepage rate q_{se} of our hemisphere is

$$q_{\mathrm{se}} = \int_0^{2\pi} \int_0^{\pi/2} V_\rho r^2 \sin(\theta)\, \mathrm{d}\theta\, \mathrm{d}\omega \tag{14}$$

where ω is the third spherical co-ordinate. Substitution of the calculated normal velocity values into (14) and elementary integrations yield

$$q_{\mathrm{se}} = 2\pi(rH - r^2/2) \tag{15}$$

To estimate the influence of cavity filling on the rate we compare q_{se} and the value of q_{sf} of a filled hemisphere. For this comparison we have to put $\Delta H = H - r$ in the classical formula[12,17] for capacity of a hemisphere-equipotential $q_{\mathrm{sf}} = 2\pi r \Delta H$, where ΔH is the head drop between sphere surface and infinity. In other words, we assume that pressures p of a filled and empty drain coincide and are atmospheric at the point B. Then from (15) $q_{\mathrm{se}} - q_{\mathrm{sf}} = \pi r^2$. Legendre polynomial expansions were used by Jankovic and Barnes[18] in studies of 3-D flows and their results can be defined as 'explicit analytic' ones. In other words, their technique does not allow to express flow characteristics in a closed form. However, 'analytic elements' they used (factually, the Legendre series) make possible accurate calculations for much more general flow patterns.

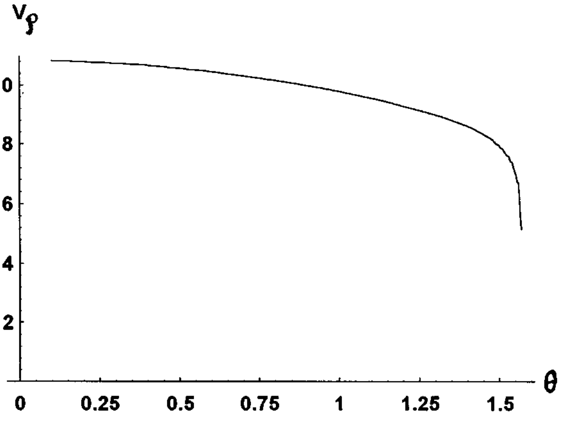


Figure 6

Note, that a filled hemisphere in this flow pattern provides minimal seepage rate in the class of arbitrary equipotential cavities of prescribed volume while the Weber disk was surmised to provide minimal rate in the class of convex filled cavities of fixed surface area.[17] As far as we know, there are no analytic shape optimizations and corresponding rate estimations for 3-D empty cavities.

It is worthy of mention, that solution for a Weber disk (which satisfies both the constant head and constant pressure conditions) under ponded surface was utilized for estimations of rate even in a free surface flow[19] that confirms usefulness of simple analytic formulae.

4. DRAINS IN HETEROGENEOUS AQUIFERS

Most soils are not homogeneous and seepage into horizontal drains in aquifers with various types of heterogeneity was intensively studied analytically.[1,4,20] In what follows, we illustrate how the known solution for flow under a concrete dam can be converted to a scheme of seepage to slot drains employing the Shulgasser[21] formula. Then we compare the seepage rate into a semicircular drain in a homogeneous aquifer (the Vedernikov formula) and in a two-layer aquifer (the formula derived by the Polubarinova-Kochina method). We restrict ourselves with the case of filled drains though extension to empty ones is straightforward.

First, consider an aquifer composed of two layers of equal thickness T but different conductivities k_2 and k_1 (Figure 7(a)). Soil surface is ponded and a horizontal (AC) or vertical (DE) slot drain (gallery) is placed on the impermeable bottom. The head drop between the soil surface and drain contour is H. The question is: how do the conductivity ratio $\chi = k_1/k_2$ and the drain length L influence the seepage rate?

From the Schulgasser[21] theorem the rate of the horizontal drain q_h is expressed through the rate q_d of flow under an impervious dam (AC) in a porous massif with 'inverted' conductivity values depicted in Figure 7(b):

$$q_\mathrm{h} = k_1 k_2 H^2 / q_\mathrm{d} \tag{16}$$

The rate of the vertical slot drain q_v is derived through the value of rate q_s under an 'equivalent' sheet piling DE (Figure 7(b)) as

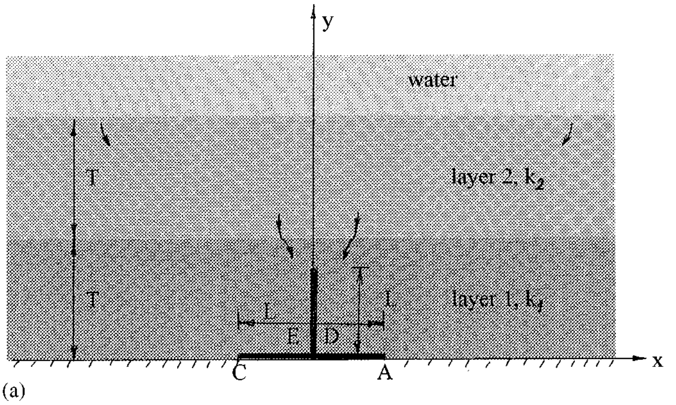
$$q_\mathrm{v} = k_1 k_2 H^2 / q_\mathrm{s} \tag{17}$$

Solutions for the two flow patterns in Figure 7(b) were developed by Polubarinova-Kochina[4] (pp. 291–308). However, in the dam problem typical length ratios L/T are high and the graphs of Polubarinova-Kochina do not fit the range usually encountered for drainage galleries. Hence, we had to recalculate the values of flow rate. In particular, the value of q_d is

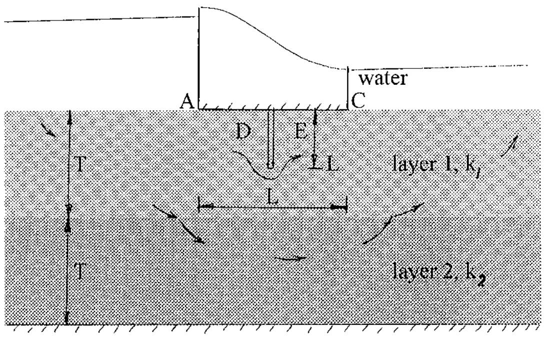
$$q_\mathrm{d} = \frac{k_1 H J_1}{2\cos(\pi\varepsilon) J},$$

$$J = \int_0^{\pi/2} \frac{\cos[2\varepsilon \arcsin(\mu \sin x)]\,\mathrm{d}x}{\sqrt{1-\mu^2\sin^2 x}}, \qquad J_1 = \int_0^{\pi/2} \frac{\cos[2\varepsilon \arcsin(\mu' \sin x)]\,\mathrm{d}x}{\sqrt{1-\mu'^2\sin^2 x}}$$

$$\varepsilon = \frac{1}{\pi}\arcsin\sqrt{\frac{1}{1+\chi}}, \qquad \mu = \tanh\frac{\pi L}{2T}, \qquad \mu' = \sqrt{1-\mu^2} \tag{18}$$



(a)



(b)

Figure 7

and the value of q_s is

$$q_s = \frac{k_1 H \tan(\varepsilon\pi)(J_3 + J_4)}{2J_2}, \qquad \text{at } L > T$$

$$= \frac{k_1 H}{J_3 + J_4}\left[\frac{\mu'^{2\varepsilon} J_5}{\cos(\varepsilon\pi)} + 0{\cdot}5(J_3 - J_4)\tan(\varepsilon\pi)\right], \qquad \text{at } L < T \tag{19}$$

where

$$J_2 = J_3 - J_4 + J_5 \frac{2\mu'^{2\varepsilon}}{\sin(\varepsilon\pi)}, \qquad J_3 = \int_0^{\pi/2} \frac{(\sqrt{1-\mu^2\sin^2 x} + \mu\cos x)^{2\varepsilon}\,\mathrm{d}x}{\sqrt{1-\mu^2\sin^2 x}}$$

$$J_4 = \int_0^{\pi/2} \frac{(\sqrt{1-\mu^2\sin^2 x} - \mu\cos x)^{2\varepsilon}\,\mathrm{d}x}{\sqrt{1-\mu^2\sin^2 x}}, \qquad J_5 = \int_0^{\pi/2} \frac{\cos(2\varepsilon x)\,\mathrm{d}x}{\sqrt{1-\mathrm{k}'^2\sin^2 x}}$$

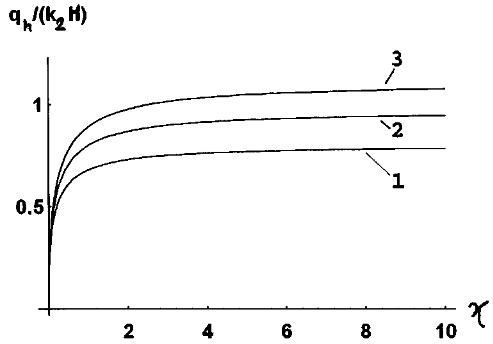


Figure 8

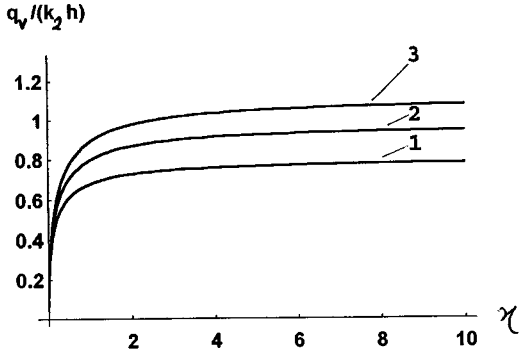


Figure 9

The value of $q_h/(k_2H)$ calculated according to (16) and (18) is shown in Figure 8 as function of χ for $L/T = 0{\cdot}05$, $0{\cdot}1$, $0{\cdot}15$ (curves 1–3, respectively). The rate $q_v/(k_2H)$ is plotted in Figure 9 according to (17) and (19) for the same L/T as in Figure 8.

In the examples above, the layers were of equal thickness. Now, we use the Fourier transform[4,11] and derive a solution for the problem of flow into a semicircular drain of small radius r placed at an impervious bottom of an aquifer composed by two layers of arbitrary conductivity $k_1, k_2, \chi = k_1/k_2$ and arbitrary thickness $d_1, d_2, d = d_1 + d_2$ (Figure 10). The soil surface is ponded and head difference between it and drain contour is H. We model the drain contour as an equipotential of a sink placed at the point O.

After some algebra (Appendix I) we evaluate the drain rate, q_{dr}:

$$\frac{q_{dr}}{k_2H} = \frac{\pi\chi}{F}, \qquad F = \ln(d_1/r) - 2\int_0^\infty f_1(\alpha)g_1(\alpha)\,d\alpha + \chi\int_0^\infty f_2(\alpha)g_2(\alpha)\,d\alpha \tag{20}$$

where

$$f_1 = 0{\cdot}5e^{-\alpha d_1}\frac{\chi p_2/p_1 - 1}{d_1\cosh(\alpha d_1) + \chi(p_2/p_1)\sinh(\alpha d_1)}$$

$$f_2 = e^{-\alpha d_1} - \frac{2\sinh(\alpha d_1)}{p_1}f_1$$

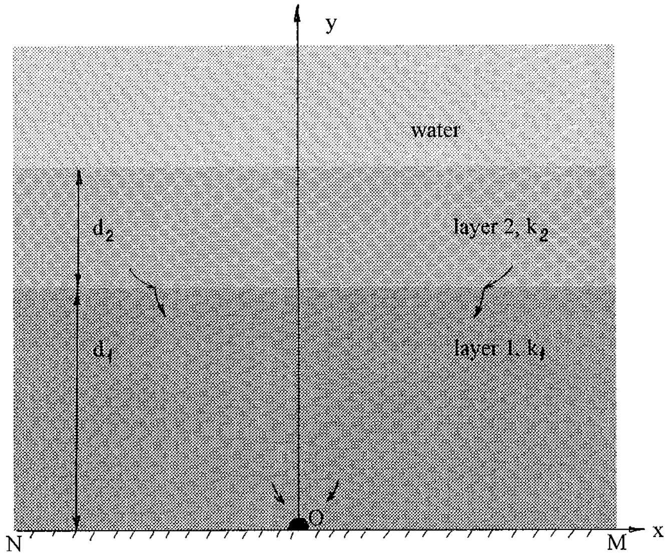


Figure 10

$$p_1 = \mathrm{e}^{-\alpha d_1} + \mathrm{e}^{-\alpha d_1 - 2\alpha d_2}, \qquad p_2 = \mathrm{e}^{-\alpha d_1 - 2\alpha d_2}$$

$$g_1 = \frac{\cosh(\alpha d_1) - \cosh(\alpha r)}{\alpha}$$

$$g_2 = (1 + \mathrm{e}^{-2\alpha d}) \frac{\sinh(\alpha d) - \sinh(\alpha d_1)}{\alpha} - (1 - \mathrm{e}^{-2\alpha d}) \frac{\cosh(\alpha d) - \cosh(\alpha d_1)}{\alpha} \tag{21}$$

At $k_2 = k_1 = k$ the Vedernikov[5] formula for the rate q^0_{dr} reads

$$q^0_{\mathrm{dr}} = \frac{\pi k H}{\operatorname{arctanh}(\sin[\pi(1 - r/d)/2])} \tag{22}$$

while the Polubarinova-Kochina[4] (p. 355) approximate formula is $q^0_{\mathrm{dr}} = (\pi k H)/(\ln[0{\cdot}5\pi r/d])$.

The rate $q_{\mathrm{dr}}/(k_2 H)$ calculated according to (20) and (21) is plotted in Figure 11 as function of χ for $r/H = 0{\cdot}05$, $d_1/H = 1$, $d_2/H = 0{\cdot}1, 1{\cdot}0, 2{\cdot}0$ (curves 1–3, respectively). *Mathematica* allowed to perform computations easily. In the limit $\chi = 1$ our calculations fit the Vedernikov formula (22) well, while the Polubarinova-Kochina one leads to some discrepancy as was the case for a filled drain in Section 2 above. Extension of the method to the case of three and more layers is straightforward.

Note, that the Fourier transforms, Fourier series expansions (developed by Kirkham, see Barua and Tiwari[22] for recent references), conformal mappings and analytic continuation principles[23] allow to take into account rigorous conjugation conditions along division lines of different media. In combination with approximate analytic description[4] of inhomogeneities it allows to complement standard numerical codes.

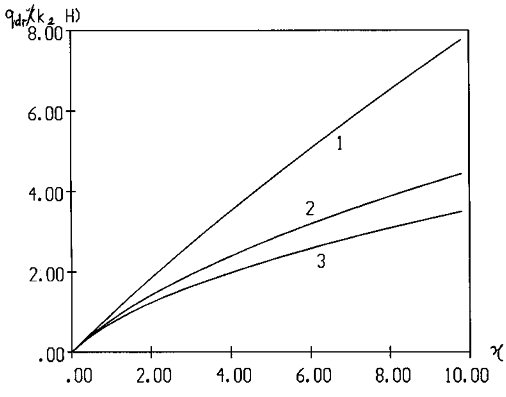


Figure 11

5. PHILIP'S MODEL OF UNSATURATED SEEPAGE

In the analysis above we studied the influence of location, the boundary condition along the drain/cavity contour and soil heterogeneity on the flow rate of saturated seepage. Here we consider unsaturated flow in terms of the quasi-linear model and dwell on effect of the drain shape on the rate value starting with the classical Wilson singularity. A comprehensive review of the model we use was done by Pullan[24] and Clothier *et al.*[25] According to the model the conductivity k varies exponentially with pressure head h, i.e. $k = k_0 e^{\alpha h}$ where constants k_0 and α are saturated conductivity and sorptivity, respectively. The governing equation for the matrix flux potential $\phi = K/\alpha$ is

$$\Delta\phi - \alpha\frac{\partial\phi}{\partial z} = 0 \tag{23}$$

where z is the vertical co-ordinate oriented downward.

Consider a single-drain placed 'near' (we discuss further what 'nearness' means) the point (0, 0) in infinite porous medium (Figure 12). Sufficiently far from the drain $\phi = k_\infty/\alpha$, i.e. the porous matrix is at constant pressure with head $h = \alpha^{-1}\ln(k_\infty/k_0)$ (obviously, $k_\infty < k_0$ to guaranty purely unsaturated seepage). The vertical and horizontal components of velocity are

$$v_z = \alpha\phi - \frac{\partial\phi}{\partial z}, \qquad v_x = -\frac{\partial\phi}{\partial x} \tag{24}$$

At infinity, we have a uniform descending flow with velocity k_∞.

Assume that along the drain contour ABC, pressure is constant and negative. Hence, the corresponding potential ϕ_c is k_c/α (obviously, $k_c < k_\infty$ to provide drainage).

Our goal is to establish how the drain rate depends on its shape and the ratio $\xi = k_c/k_\infty$. For this purpose similarly to the Forchheimer scheme[2] for saturated flow and Bystrov,[26] Philip[27] solution for (23) we model the drain as a single source of strength q_u placed at the origin of coordinates and follow the Philip[27] procedure of definition of characteristic drain sizes. The flow

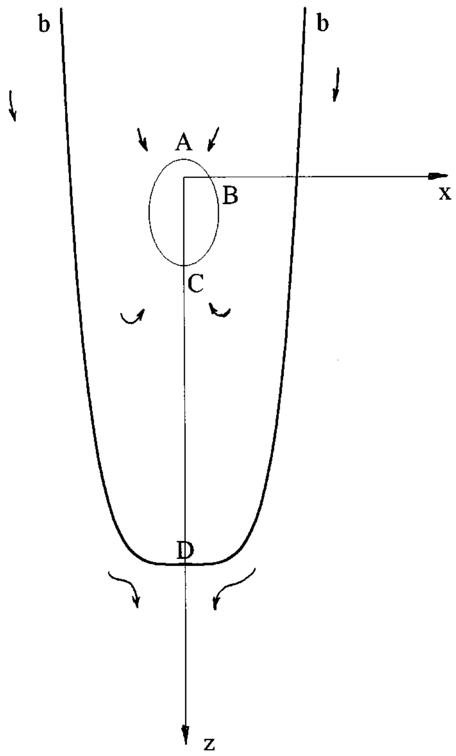


Figure 12

potential is[28]

$$\phi = \frac{k_\infty}{\alpha} - \frac{q_u}{2\pi} e^{Z} K_0(\sqrt{X^2 + Z^2}) \tag{25}$$

where $Z = \alpha z$, $X = \alpha x$ are non-dimensional co-ordinates and K_0 is the modified Bessel function of second kind and zero order. Introduce non-dimensional variables $\Phi = \phi\alpha/k_\infty$, $Q = \alpha q_u/(2\pi k_\infty)$.

The drain shape is determined from (25) putting $\phi = k_c/\alpha$ as

$$1 - \xi = Q e^{Z} K_0(\sqrt{X^2 + Z^2}) \tag{26}$$

The co-ordinates Z_C and Z_A of the points C and A are found directly from (25) at $X = 0$. The co-ordinates (X_B, Z_B) of the point B are determined from the condition $X'(Z) = 0$ along ABC as a root of the system

$$\begin{gathered} 1 - \xi = Q e^{Z} K_0(\sqrt{X^2 + Z^2}) \\ \frac{K_0(\sqrt{X^2 + Z^2})}{K_1(\sqrt{X^2 + Z^2})} = \frac{Z}{\sqrt{X^2 + Z^2}} \end{gathered} \tag{27}$$

where K_1 is the Bessel function of first order.

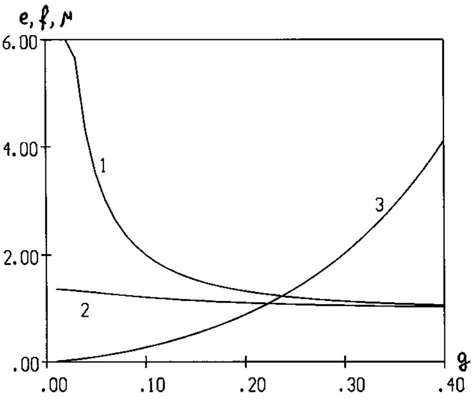


Figure 13

Both (26) and (27) are solved using standard *Mathematica* routines.[29] We define the horizontal and vertical sizes (recall that they are nondimensional) of the drain as $b = X_B$ and $a = (Z_C - Z_A)/2$, correspondingly. We define position of the centre of the drain as $c = (Z_C + Z_A)/2$. Calculations show that ABC is an oval which can be approximated as an ellipse and we determine the cross-sectional area of the drain as $S = \pi ab$. The oval ratio we define as $e = a/b$ and the value of $f = c/z_B$ characterizes deviation of the oval from an equivalent ellipse. Figure 13 shows the values of e, f and $\mu = Q/\sqrt{S}$ (curves 1–3, correspondingly) as functions of $g = 1 - \xi$ at $Q = 1$. Note, that at sufficiently high values of ϕ_c the corresponding drain 'centre' lies far below the sink. It means the isobaric contour seeps mostly through its upper part.

The drain captures a part of the descendant flow and the line b–b in Figure 12 is a separatrice bisecting the flow. The stagnation point D can be found according to (23) from the condition $v_z(0, z) = 0$ that in non-dimensional variables is reduced to

$$\mathrm{e}^{Z}[K_1(Z) - K_0(Z)] = 2/Q \tag{28}$$

The root Z_D of (2) is shown in Figure 14 as function of Q. Note, that in 2-D saturated flow for a similar pattern of a single pumping well in uniform flow (Bear,[30] pp. 323–324) the distance between the well and stagnation point is $q/(2\pi v_\infty)$ where q is the well rate and v_∞ is the velocity of the uniform flow. As we have mentioned, the case of an isobaric circle was studied by Lei[9] though he did not distinguish explicitly the separatrice and the drain catchment area.

Emphasize that all geometrical sizes of the drain evaluated are the same for the case of a cavity modeled by a single source in infinite medium[27] since potentials of the two flows differ only in a constant (ϕ_∞) though flow nets and velocity distributions clearly differ.

6. OPTIMAL SHAPE DESIGN

Since the classical studies of ancient Greeks, Saint-Venant, and Rayleygh shape optimization and corresponding isoperimetric inequalities are investigated in many applications,[17,31,32] in particular, for seepage flows.[33,34] The general statement is clear: what form of the boundary (or its part)

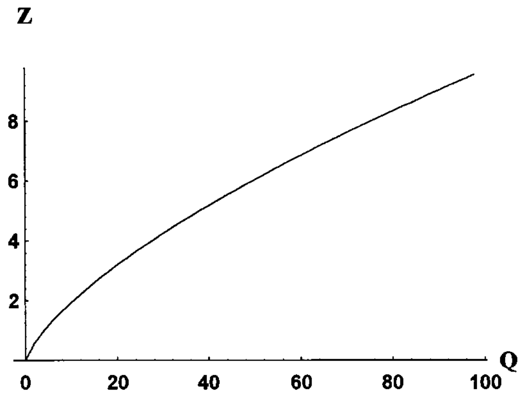


Figure 14

of the flow domain provides extreme (minimal or maximal) flow characteristic (say, rate, uplift force, wetted area, etc.) at prescribed isoperimetric restrictions (for example length, area or volume). In the section devoted to 3-D saturated flows we have mentioned one of the results that dates back to Poincaré (a sphere in an unrestricted aquifer provides minimal rate in the class of equipotential bodies of prescribed volume). In what follows, we treat one of the problems of this kind for 2-D unsaturated flow in terms of the quasi-linear model. Unlike the cases above we consider an irrigation cavity which wets the surrounding porous matrix. To our knowledge, in the analytic solutions[35,36] for a single-cavity wetting unrestricted soil only circular and elliptical contours were investigated.

Let us study seepage from the contour $\Gamma(ABCA)$ of constant moisture (pressure) with potential $\phi = \phi_c$ (Figure 12). At infinity $\phi = \phi_\infty$, $\phi_\infty < \phi_c$. Γ confines the domain Ω of area S. Outside the cavity potential satisfies the advective dispersion equation (22) with moisture velocities (23). Designate the total seepage rate as q_u.

We want to determine the shape which provides minimal q_u (a criterion) at prescribed S (an isoperimetric restriction), ϕ_c, ϕ_∞. First, consider the case of arbitrary cavities (though with sufficiently smooth contours).

It is well known that minimum should satisfy a necessary condition of optimality (for a function of one variable $C(c)$ this states $\mathrm{d}C/\mathrm{d}c = 0$ where C is the criterion and c is the control variable) and a sufficient one ($\mathrm{d}^2C/\mathrm{d}c^2 > 0$). In our case cavity shape is the control function and the criterion depends on infinite number of variables. It calls for subtle methods of boundary variations one of which is presented below. In what follows we derive a necessary condition for minimum:

$$J(\Omega) = q_u = \oint_\Gamma v_n(s)\,\mathrm{d}s = -\oint_\Gamma \nabla\phi\cdot\mathbf{n}\,\mathrm{d}s + \alpha\oint_\Gamma \phi n_z\,\mathrm{d}s \tag{29}$$

where v_n designates the normal component of velocity, $\mathbf{n} = (n_z, n_x)$ is the inward normal (from the cavity to soil), s is the arc length of the contour (counterclockwise).

Let $\boldsymbol{\tau}(s) = (\tau_z, \tau_x)$ be the tangential vector directed counterclockwise at s. Then, $\tau_z = -n_x$, $\tau_x = n_z$.

We can easily see that

$$\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} = \frac{1}{R}\boldsymbol{\tau} \tag{30}$$

hence, in turn

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\mathbf{s}} &= \left(-\frac{\mathrm{d}n_x}{\mathrm{d}s}, \frac{\mathrm{d}n_z}{\mathrm{d}s}\right)\\ &= -\frac{1}{R}\mathbf{n}\end{aligned} \tag{31}$$

where R denotes the radius of curvature at s. On Γ the formula of integration by parts reads

$$\oint_{\Gamma} f'(s)g(s)\,\mathrm{d}s = \oint_{\Gamma} f(s)g'(s)\,\mathrm{d}s \tag{32}$$

Let us introduce a variation of Γ. Let $\rho(s)$ be a smooth function of s. Let ε be a number; its absolute value is small enough. We place segment $\varepsilon\rho(s)$ on the normal $\mathbf{n}$ at s such that positive $\varepsilon\rho(s)$ lies on the normal $\mathbf{n}$. If $|\varepsilon|$ is small enough, the end points of the segments will form a closed curve Γ_ε which and Γ_∞ enclose a new domain Ω_ε. When we consider the following boundary-value problem:

$$\Delta\phi^\varepsilon = \alpha\frac{\partial\phi^\varepsilon}{\partial z} \quad ((z, x) \in \Omega_\varepsilon) \tag{33}$$

$$\phi^\varepsilon = \phi_\mathrm{c}(\text{const.}) \quad ((z, x) \in \Gamma_\varepsilon) \tag{34}$$

$$\phi^\varepsilon = \phi_\infty \quad ((z, x) \in \Gamma_\infty) \tag{35}$$

we can easily find that the first variation Θ of ϕ defined by

$$\phi^\varepsilon - \phi = \varepsilon\Theta + o(\varepsilon) \tag{36}$$

is the solution of

$$\Delta\Theta = \alpha\frac{\partial\Theta}{\partial z} \quad (\text{in } \Omega) \tag{37}$$

$$\Theta = -\frac{\partial\phi}{\partial n}\rho \quad (\text{on } \Gamma) \tag{38}$$

$$\Theta = 0 \quad (\text{at } \Gamma_\infty) \tag{39}$$

On the other hand, we see that the corresponding $\mathbf{n}_\varepsilon$ is given by

$$\mathbf{n}^\varepsilon = \mathbf{n} + (-\varepsilon\rho'(s)\boldsymbol{\tau}) + o(\varepsilon) \tag{40}$$

through geometrical inspection. Similarly, we obtain

$$\mathrm{d}s^\varepsilon = \left(1 + \frac{\varepsilon\rho}{R} + o(\varepsilon)\right)\mathrm{d}s \tag{41}$$

Objective functional J^ε for ϕ^ε is given by

$$J^\varepsilon = -\oint_{\Gamma_\varepsilon} \operatorname{grad} \phi^\varepsilon \cdot \mathbf{n}^\varepsilon \mathrm{d}s^\varepsilon + \alpha \oint_{\Gamma_\varepsilon} \phi^\varepsilon n_z^\varepsilon \,\mathrm{d}s^\varepsilon \tag{42}$$

The first term on the right-hand side of (42) is transformed as follows:

$$\begin{aligned}\oint_{\Gamma_\varepsilon} \operatorname{grad} \phi^\varepsilon \cdot \mathrm{d}s^\varepsilon &= \oint_{\Gamma_\varepsilon} \operatorname{grad}(\phi + \varepsilon\Theta + o(\varepsilon)) \cdot \mathbf{n}^\varepsilon \,\mathrm{d}s^\varepsilon \\ &= \oint_{\Gamma} \operatorname{grad} \phi \cdot \mathbf{n} \,\mathrm{d}s \\ &\quad + \varepsilon \oint_{\Gamma} \left\{ \frac{1}{R} \operatorname{grad} \phi \cdot \mathbf{n} + \left(\frac{\partial^2 \phi}{\partial z^2} n_z^2 + 2 \frac{\partial^2 \phi}{\partial z \partial x} n_z n_x + \frac{\partial^2 \phi}{\partial x^2} n_x^2 \right) \right\} \mathrm{d}s \\ &\quad + \varepsilon \oint_{\Gamma_\varepsilon} \operatorname{grad} \Theta \cdot \mathbf{n} \,\mathrm{d}s - \varepsilon \oint_{\Gamma_\varepsilon} \rho'(s) \operatorname{grad} \phi \cdot \boldsymbol{\tau} \,\mathrm{d}s + o(\varepsilon) \end{aligned} \tag{43}$$

where we used (40) and (41).

Introduce an adjoint variable p_a as the solution of the following boundary value problem:

$$\Delta p_\mathrm{a} + \alpha \frac{\partial p_\mathrm{a}}{\partial z} = 0 \quad \text{(in soil)} \tag{44}$$

$$p_\mathrm{a} = 1 \quad \text{(at the cavity boundary)} \tag{45}$$

$$p_\mathrm{a} = 0 \quad \text{(at infinity)} \tag{46}$$

Thanks to Green's formula, we can calculate as follows:

$$\begin{aligned}\int_\Omega (\Theta \Delta p_\mathrm{a} - p_\mathrm{a} \Delta\Theta) \,\mathrm{d}a &= \oint_{\Gamma + \Gamma_\varepsilon} \left(p_\mathrm{a} \frac{\partial \Theta}{\partial n} - \Theta \frac{\partial p_\mathrm{a}}{\partial n} \right) \mathrm{d}s \\ &= \oint_\Gamma \operatorname{grad} \Theta \cdot \mathbf{n} \,\mathrm{d}s + \oint_\Gamma \frac{\partial \phi}{\partial n} \frac{\partial p_\mathrm{a}}{\partial n} \rho \,\mathrm{d}s \end{aligned} \tag{47}$$

where we used (38), (39), (45) and (46). Hence, using (37), (44), (38) and (39), we obtain

$$\begin{aligned}\oint_\Gamma \operatorname{grad} \Theta \cdot \mathbf{n} \,\mathrm{d}s + \oint_\Gamma \frac{\partial \phi}{\partial n} \frac{\partial p_\mathrm{a}}{\partial n} \rho \,\mathrm{d}s &= \int_\Omega (\Theta \Delta p_\mathrm{a} - p_\mathrm{a} \Delta \Theta) \,\mathrm{d}a \\ &= -\int_\Omega \left(\alpha \frac{\partial p_\mathrm{a}}{\partial z} \Theta + \alpha p_\mathrm{a} \frac{\partial \Theta}{\partial z} \right) \mathrm{d}a \\ &= -\int_\Omega \operatorname{div}(\boldsymbol{\alpha} p_\mathrm{a} \Theta) \,\mathrm{d}a \\ &= \oint_\Gamma p_\mathrm{a} \Theta \boldsymbol{\alpha} \cdot \mathbf{n} \,\mathrm{d}s + \oint_{\Gamma_\infty} p_\mathrm{a} \Theta \boldsymbol{\alpha} \cdot \mathbf{n} \,\mathrm{d}s \\ &= -\oint_\Gamma \alpha \frac{\partial \phi}{\partial n} \rho n_z \,\mathrm{d}s \end{aligned} \tag{48}$$

where $\boldsymbol{\alpha} = (\alpha, 0)$.

On the other hand, by integrating by parts, we obtain

$$-\oint_\Gamma \rho'(s) \operatorname{grad} \phi \cdot \boldsymbol{\tau} \, ds = \oint_\Gamma \rho(s) \frac{\partial}{\partial s} (\operatorname{grad} \phi \boldsymbol{\tau}) \, ds$$

$$= \oint_\Gamma \left\{ \left(\frac{\partial}{\partial s} \operatorname{grad} \phi \right) \cdot \boldsymbol{\tau} - \frac{1}{R} \operatorname{grad} \phi \cdot \mathbf{n} \right\} \rho \, ds \tag{49}$$

Substituting (48) and (49) into (43), we obtain

$$\oint_{\Gamma_\varepsilon} \operatorname{grad} \phi^\varepsilon \cdot ds^\varepsilon = \oint_\Gamma \operatorname{grad} \phi \cdot \mathbf{n} \, ds + \varepsilon \oint_\Gamma \left(\frac{\partial^2 \phi}{\partial z^2} n_z^2 + 2 \frac{\partial^2 \phi}{\partial z \partial x} n_z n_x + \frac{\partial^2 \phi}{\partial x^2} n_x^2 \right) \rho \, ds$$

$$- \varepsilon \oint_\Gamma \left(\frac{\partial \phi}{\partial n} \frac{\partial p_a}{\partial n} \rho + \alpha \frac{\partial \phi}{\partial n} n_z \rho \right) ds$$

$$+ \varepsilon \oint_\Gamma \left(\frac{\partial}{\partial s} \operatorname{grad} \phi \right) \cdot \boldsymbol{\tau} \rho + o(\varepsilon) \tag{50}$$

Similarly, we can see that

$$\oint_{\Gamma_\varepsilon} \phi^\varepsilon n_z^\varepsilon \, ds^\varepsilon = \oint_\Gamma \phi n_z \, ds + \varepsilon \oint_\Gamma \left(\frac{1}{R} \phi \rho n_z + \rho \frac{\partial}{\partial s} (\phi \tau_z) \right) ds$$

$$= \oint_\Gamma \phi n_z \, ds + o(\varepsilon) \tag{51}$$

since $(\partial \phi / \partial s)$ vanishes on Γ. From (50) we see that

$$\oint_{\Gamma_\varepsilon} \operatorname{grad} \phi^\varepsilon \cdot ds^\varepsilon - \oint_\Gamma \operatorname{grad} \phi \cdot \mathbf{n} \, ds = -\varepsilon \oint_\Gamma \left(\frac{\partial \phi}{\partial n} \frac{\partial p_a}{\partial n} \rho + \alpha \frac{\partial \phi}{\partial n} n_z \rho \right) ds$$

$$+ \varepsilon \oint_\Gamma \left(\frac{\partial^2 \phi}{\partial z^2} n_z^2 + 2 \frac{\partial^2 \phi}{\partial z \partial x} n_z n_x + \frac{\partial^2 \phi}{\partial x^2} n_x^2 \right) \rho \, ds$$

$$+ \varepsilon \oint_\Gamma \left(\frac{\partial^2 \phi}{\partial z^2} n_z^2 + - 2 \frac{\partial^2 \phi}{\partial z \partial x} n_z n_x + \frac{\partial^2 \phi}{\partial x^2} n_x^2 \right) \rho \, ds + o(\varepsilon)$$

$$= -\varepsilon \oint_\Gamma \left(\frac{\partial \phi}{\partial n} \frac{\partial p_a}{\partial n} \rho + \alpha \frac{\partial \phi}{\partial n} n_z \rho \right) ds + \varepsilon \alpha \oint_\Gamma \frac{\partial \phi}{\partial z} \rho \, ds + o(\varepsilon) \tag{52}$$

Hence, if we define δJ by

$$J^\varepsilon - J = \varepsilon \delta J + o(\varepsilon) \tag{53}$$

we observe that

$$\delta J = -\oint_\Gamma \left(\frac{\partial \phi}{\partial n} \frac{\partial p_a}{\partial n} + \alpha \frac{\partial \phi}{\partial n} n_z - \alpha \frac{\partial \phi}{\partial z} \right) \rho \, ds \tag{54}$$

Since admissible cavities must satisfy

$$\int_{\Omega^c} \mathrm{d}a = S \tag{55}$$

where S is the given area of cross-section of the cavities, we have

$$\oint_{\Gamma} \rho \, \mathrm{d}s = 0 \tag{56}$$

If Ω attains minimum q_u, δJ must vanish for every ρ which satisfies (56). Thus, we obtain the following necessary condition of minimum:

If the cavity boundary Γ attains a minimum q_u, then there exists a constant λ (Lagrange multiplier) such that

$$\frac{\partial \phi}{\partial n}\frac{\partial p_a}{\partial n} + \alpha n_z \frac{\partial \phi}{\partial n} - \alpha \frac{\partial \theta}{\partial z} = \lambda \tag{57}$$

holds at every point on Γ.

The known sink–source solutions of (23) and the solution of (44) are given by infinite series expansions in terms of modified Bessel functions; it is difficult to test the condition (57) for these solutions. However, (54) gives the gradient of the objective function; (54) can be used for numerical calculation of the optimal shape. As for more general and prototype shape optimization, the readers can refer Fujii.[32]

Preliminary computations based on the Concer[35] solution and McLachlan[37] asymptotics (small values of the ellipse aspect ratio) showed that starting with a circle and preserving cross-sectional area of a cavity we do can improve the criterion, i.e. minimize the rate (Appendix II). Nevertheless, matching of the rigorous necessary condition and the corresponding condition for an optimal ellipse (in some integral sense[31]) is still an unresolved problem.

7. CONCLUSION AND PERSPECTIVES

We addressed a number of problems for saturated and unsaturated, 2-D and 3-D flow patterns in homogeneous and layered porous matrixes. Delving in the old books we found a number of solutions which seemed to be ponderous at the time they were developed. However, the progress in computer treatment of analytic expressions makes them competitive with standard numerical procedures. In this way we applied the complex analysis, Fourier expansions and transform, optimal shape design methods and illustrated the simple and tractable solutions utilizing *Mathematica*. Of most interest for us are optimization problems for seepage flows that can be treated analytically. Note, that by virtue of close analogy between the quasi-linear model for unsaturated seepage and advective dispersion model of neutral contaminant transport in subsurface we plan to employ optimization methods in estimations of tracer distributions.

Obviously, the solutions are derived under severe simplifications about the flow and medium and can serve as test, ‘back-of-an-envelope’ procedures in implementation of FDM-FEM codes to ‘real-world’ problems.

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APPENDIX I

In what follows we describe in details derivations of the total rate for the flow in Figure 10 (Section 4).

Factually, we use standard technique of Polubarinova-Kochina (Fourier integral presentations). However, in the book of Olejnik (1979) which summarizes the problems solved by this method we could not find an analogous solution. Olejnik (and the authors he cites) have used some special tricks (series expansions) to derive the final formulae for total rate (it was necessary in the pre-Mathematica years). We do not need any additional assumptions and perform integration without any simplifications.

Thus, we search for two analytic functions $V_1 = u_1 - \mathrm{i}v_1 = \mathrm{d}w_1/\mathrm{d}z$ and $V_2 = u_2 - \mathrm{i}v_2 = \mathrm{d}w_2/\mathrm{d}z$ where w_1 and w_2 are complex potentials within the two layers, V_1 and V_2 are complex conjugated velocities, $u_{1,2}$ and $v_{1,2}$ are horizontal and vertical components of the velocity. Along MO we assume stream function $\psi = 0$ and $\psi = -q_{\mathrm{dr}}$ along NO. Since the drain is filled we set potential $\phi = 0$ along its contour and $\phi = -H$ along the flooded surface.

Like in the vortex case treated by Polubarinova-Kochina (1977) we present V_1 as

$$V_1 = -\frac{q_{\mathrm{dr}}}{\pi z} + \int_0^\infty [A_1(\alpha)\mathrm{e}^{\mathrm{i}\alpha z} + B_1(\alpha)\mathrm{e}^{-\mathrm{i}\alpha z}]\,\mathrm{d}\alpha \tag{58}$$

The first summand in equation (58) corresponds to the half-sink modelling our drain. Analogously, in the upper layer

$$V_2 = \int_0^\infty [A_2(\alpha)\mathrm{e}^{\mathrm{i}\alpha z} + B_2(\alpha)\mathrm{e}^{-\mathrm{i}\alpha z}]\,\mathrm{d}\alpha \tag{59}$$

(the singular term is omitted because drain is located in the first layer).

The complex coefficients $A_1 = a_{r1} + \mathrm{i}a_{i1}$, $B_1 = b_{r1} + \mathrm{i}b_{i1}$, $A_2 = a_{r2} + \mathrm{i}a_{i2}$, $B_2 = b_{r2} + \mathrm{i}b_{i2}$ have to be found. To derive the eight unknown values we use the boundary conditions. Namely, along the horizontal bottom $v_1 = 0$. Hence, equating the imaginary part in (58) to zero we come to $a_{r1} = b_{r1}$ and $a_{i1} = -b_{i1}$. Along the ponded surface $u_2 = 0$. Therefore, from (59) we derive $b_{r2} = -a_{r2}\mathrm{e}^{-2\alpha d}$ and $b_{i2} = -a_{i2}\mathrm{e}^{-2\alpha d}$.

Along the contact line of the layers two conditions hold. First, $u_1 = \chi u_2$ (jump in tangential velocities). Present the singular term in (58) as

$$1/z = -\mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}\alpha z}\,\mathrm{d}\alpha$$

Equate the real part of (58) and real part of (59) multiplied by χ. After elementary calculations we come to

$$a_{i1}(\mathrm{e}^{\alpha d_1} + \mathrm{e}^{-\alpha d_1}) + \frac{q_{\mathrm{dr}}}{\pi}\mathrm{e}^{-\alpha d_1} = \chi a_{i2}(\mathrm{e}^{-\alpha d_1} + \mathrm{e}^{-\alpha d_1 - 2\alpha d_2})$$

$$a_{r1}(\mathrm{e}^{\alpha d_1} + \mathrm{e}^{-\alpha d_1}) = \chi a_{r2}(\mathrm{e}^{-\alpha d_1} + \mathrm{e}^{-\alpha d_1 - 2\alpha d_2})$$

The second conjugation condition along the interface between two layers states that $v_1 = v_2$ (continuity of normal velocities). Therefore, equating imaginary parts of (58) and (59) along this line we obtain

$$a_{r1}(\mathrm{e}^{-\alpha d_1} - \mathrm{e}^{\alpha d_1}) = a_{r2}(\mathrm{e}^{-\alpha d_1} + \mathrm{e}^{-\alpha d_1 - 2\alpha d_2})$$

$$\frac{q_{\mathrm{dr}}}{\pi}\mathrm{e}^{-\alpha d_1} + a_{i1}(\mathrm{e}^{-\alpha d_1} - \mathrm{e}^{\alpha d_1}) = a_{i2}(\mathrm{e}^{-\alpha d_1} + \mathrm{e}^{-\alpha d_1 - 2\alpha d_2})$$

It follows immediately, that $a_{r1} = b_{r1} = a_{r2} = b_{r2} = 0$. Note, that this result could be obtained from some simple symmetry reasons. However, we carried out all derivations to illustrate the procedure. For a_{i1} and a_{i2} we have a system of linear equations which yields

$$a_{i1} = \frac{q_{dr}}{2\pi} e^{-\alpha d_1} \frac{\chi p_2/p_1 - 1}{\cosh(\alpha d_1) + \chi p_2/p_1 \sinh(\alpha d_1)} \tag{60}$$

$$a_{i2} = \frac{q_{dr} e^{-\alpha d_1}}{\pi p_1} - \frac{2a_{i1} \sinh(\alpha d_1)}{p_1}$$

Thus, we derived completely our solution

$$V_1 = -\frac{q_{dr}}{\pi z} - 2\int_0^\infty a_{i1} \sin(\alpha z)\, d\alpha$$

$$V_2 = i\int_0^\infty a_{i2}[\cos(\alpha z)(1 + e^{-2\alpha d}) + i \sin(\alpha z)(1 - e^{-2\alpha d})]\, d\alpha \tag{61}$$

where a_{i1} and a_{i2} are defined by (60).

However, to determine the total rate we have to circumvent a technical pitfall that is not perfectly explained in textbooks (say, in the books by Oleynik and Polubarinova-Kochina). Namely, we perform integration of (61) and derive the complex potentials

$$w_1 = -\frac{q_{dr}}{\pi} \ln z + 2\int_0^\infty a_{i1} \cos(\alpha z)\, d\alpha + w_{10} \tag{62}$$

$$w_2 = i\int_0^\infty a_{i2}\left[\frac{\sin(\alpha z)}{\alpha}(1 + e^{-2\alpha d}) - \frac{\cos(\alpha z)}{\alpha}(1 - e^{-2\alpha d})\right] d\alpha + w_{20}$$

Now, we employ the boundary conditions for ϕ and ψ to find the constants of integration w_{10}, w_{20}. At the point $z = ir$ the first complex potential $w_1 = 0$. Hence, from the first equation (62)

$$w_{10} = \frac{q_{dr}}{\pi} \ln(ir) - 2\int_0^\infty a_{i1} \frac{\cos(\alpha i r)}{\alpha}\, d\alpha$$

In the first layer the potential at the point $z = id_1$ is

$$\phi_1 = -\frac{q_{dr}}{\pi} \ln(d_1/r) + 2\int_0^\infty a_{i1} \frac{\cosh(\alpha d_1) - \cosh(\alpha r)}{\alpha}\, d\alpha \tag{63}$$

At the point $z = id$ the second complex potential is $w_2 = -iq_{dr}/2 - k_2 H$. Therefore, we can determine w_{20} and the potential in the second layer is

$$\phi_2 = -k_2 H + \int_0^\infty a_{i2}\left[\frac{\sinh(\alpha d)}{\alpha}(1 + e^{-2\alpha d}) - \frac{\cosh(\alpha d)}{\alpha}(1 - e^{-2\alpha d})\right] d\alpha$$

$$+ \int_0^\infty a_{i2}\left[\frac{-\sinh(\alpha d_1)}{\alpha}(1 + e^{-2\alpha d}) - \frac{\cosh(\alpha d_1)}{\alpha}(1 - e^{-2\alpha d})\right] d\alpha \tag{64}$$

Now, continuity of the pressure (head) along the line $y = id_1$ yields $\phi_1 = \chi\phi_2$ which we apply at the point $z = id_1$. In other words, we write a linear combination of (63) and (64) with χ as a coefficient. It allows to get the expression $q_{dr}/H = f(k_2, d_1, d_2, r, \chi)$, i.e. formula (19).

APPENDIX II

Consider seepage from elliptical cavities wetting surrounding dry soil. We show how the Concer[35] solution can be implemented for optimization in the class of ellipses which allow to follow the main trend in rate value under shape variations. Note, that considering narrower classes of curves is standard to approach the optimum. In this case the objective can be often approximated with good accuracy.[38] Thus, let the feeding contour $ABCA$ in Figure 12 is an ellipse with axes a, b, focus distance $c=\sqrt{a^2-b^2}$, cross-sectional area $S=\pi ab$ and rate q_{e}. Hence, in the class of ellipses under study the optimization problem is reduced to search for the axis ratio b/a providing minimum q_e at fixed S.

First, note that the Philip[36] solution coincides (up to notations) with the Concer[35] solution. In particular, Philip's formula (38) for non-dimensional total rate from a cylindrical cavity being multiplied by $2\pi r$, where r is the cylinder radius, brings just the same result as formula (27) of Concer.[35]

The value of q_{e} for an elliptical cavity we calculate after slight modification of the Concer (1959) solution:

$$q_e = \omega\pi \sinh s_0$$

$$\sum_{n=0}^{\infty} E_n Fek_n(s_0,\ -g)[(-1)^n \sum_{r=0}^{\infty} (-1)^r A_{2r}^{2n}(I_{2r+1}(\omega\cosh s_0) + I_{2r-1}(\omega\cosh(s_0))$$

$$+(-1)^n \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{2n+1}(I_{2r+2}(\omega\cosh s_0) + I_{2r}(\omega\cosh(s_0))]$$

$$+2\pi \sum_{n=0}^{\infty} E_n Fek_n'(s_0,\ -g)[(-1)^n \sum_{r=0}^{\infty} (-1)^r A_{2r}^{2n} I_{2r}(\omega\cosh s_0)$$

$$+(-1)^n \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{2n+1} I_{2r+1}(\omega\cosh s_0)] \tag{65}$$

where $s_0=\operatorname{arctanh}(b/a)$, $g=(\alpha c/4)^2$, $\omega=\sqrt{g}/2$. Note that our g is q of Concer. Coefficients E_n in (65) are found from the boundary condition along the cavity contour as

$$E_n = \frac{\phi_{\mathrm{c}}\beta_n}{Fek_n(s_0,\ -g)},$$

$$\beta_n = \frac{\int_0^{2\pi} ce_n(t,\ -g)\mathrm{e}^{-\alpha\cos t}\,\mathrm{d}t}{\int_0^{2\pi} ce_n^2(t,\ -g)\,\mathrm{d}t}$$

Definitions of the Mathieu Fek, ce and Bessel I functions are from McLachlan.[37] Note, that we assume coefficient normalization according to Section 2.11 of McLachlan.

We considered ellipses with nearly equal axes ($g\to 0$). In this case we used the asymptotic representations for coefficient from Section 3.33 of McLachlan. We calculated series expansions of Fek functions from Section 8.30 of McLachlan. We also used series expansions for ce_0, ce_1, ce_2, se_1, se_2 and their derivatives form Sections 2·13, 2·14, 2·150 of McLachlan. Same expansions we used to calculate integrals in formulae for β_{n}. Computations showed decrease in q_{e} at fixed S with increase of a/b. However, the following questions are still open. First, it is unclear, whether improve of the criterion can be tracked further in a/b using asymptotics of small g. Intuitively it is

not clear, how prolonged will be the optimal ellipse. Nevertheless, we surmise existence of a unique global optimum in the class of elliptic curves, i.e. a unique a/b value providing the minimum of q_e. This guess is based on our experience of optimization in the case of saturated flows. Factually, existence of a nontrivial minimum can be explained just as in minimization of the rate in Section 2: very prolate and very oblate cavities of constant area ('needles' placed along or perpendicular to the vertical axes) give infinitely high rates. Hence, an 'intermediate' ellipse has to provide minimum (though uniquiness of the extremum is not obvious and counter-intuitive examples were obtained just in the class of ellipses[39]). Second, it may be better to use not the asymptotics but implement the general procedure for determination of coefficients[37] or build-in procedures for Mathieu functions from 3.0 version of *Mathematica*.

However, analysis of the Concer solution was instructive in the following sense. Namely, instead of attempts to satisfy the constant pressure conditions along a prescribed contour (as in the Concer and Philip solutions) it seems to be interesting to meet the necessary condition of optimality from our Section 6 using the approach from Section 5. Namely, the first term in Concer's expansion (11) is just a point source solution (Wilson's singularity). Let us retain the second term $K_1(r)\cos(\theta)$. Then we can operate with the coefficient E_1 and try to minimize the rate from the corresponding cavity (this rate is easily determined from E_0 and E_1). The contour of this $E_0 - E_1$ cavity we can define according to the condition $\phi = \phi_c$. In other words, fixing the cross-sectional area we have to solve an equation similar to (26) from Section 5 of our paper. The procedure can be extended further for E_2, E_3, etc. This approach was employed for optimization in the case of the Poisson equation.[40]

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